

Symplectic geometry — Mock exam

Duration: 2.5 hours

Conventions. As in the lecture, Hamiltonian vector fields are defined by $\iota_{X_f}\omega = df$, and $\{f, g\} = X_f(g)$.

Exercise 1. On $T^*\mathbb{R}^3$ with coordinates $q_1, q_2, q_3, p_1, p_2, p_3$ and $\omega = \sum_i dq_i \wedge dp_i$, let

$$f = q_1 p_2 + \frac{1}{2} p_3^2, \quad g = q_2 p_3.$$

Calculate $L_{X_f}\omega$, $X_f(g)$, $\{f, g\}$ and $[X_f, X_g]$.

Exercise 2. Let L be a Lie-Rinehart algebra over A with anchor ρ . An A -submodule $L' \subset L$ is called a Lie-Rinehart subalgebra of L if it is involutive, i.e. $[x, y] \in L'$ for all $x, y \in L'$. (Then L' , with the restricted bracket and anchor $\rho|_{L'}$, is again a Lie-Rinehart algebra over A .)

Let M be a smooth manifold. Show that the time-dependent vector fields $\mathfrak{X}_t(M) \subset \mathfrak{X}(\mathbb{R} \times M)$ form a Lie-Rinehart subalgebra of the Lie-Rinehart algebra $\mathfrak{X}(\mathbb{R} \times M)$ over $C^\infty(\mathbb{R} \times M)$.

Exercise 3.

1. Let M be a manifold of dimension > 2 and $\theta_1, \theta_2 \in \Omega^1(M)$. Show that $\theta_1 \wedge \theta_2$ is never symplectic.
2. Show that the form $dx_1 \wedge dx_2 + dx_1 \wedge dy_2 + dy_1 \wedge dx_2 + dy_1 \wedge dy_2$ on $\mathbb{R}^4$ is not symplectic.
3. Show that the form $e^{x_1}(dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$ on $\mathbb{R}^4$ is not symplectic.
4. Show that $S^1 \times S^3$ admits no symplectic form.

Exercise 4. Let Q be a smooth manifold, $\theta \in \Omega^1(T^*Q)$ the canonical 1-form and $\pi : T^*Q \rightarrow Q$ the projection. Show that for any closed 2-form $\beta \in \Omega^2(Q)$, the form $d\theta + \pi^*\beta \in \Omega^2(T^*Q)$ is symplectic.

Exercise 5. On $M = \mathbb{R}^6$ with coordinates $x_1, \dots, x_6$ let $\beta = x_1 dx_2 \wedge dx_3 + e^{x_4} dx_5 \wedge dx_6$ and $\omega = d\beta$.

1. Show that $d\omega = 0$ and that the map $\iota_{\bullet}\omega : TM \rightarrow \Lambda^2 T^*M$ is injective. Is ω symplectic?
2. Calculate $\omega \wedge \omega$.
3. Let $\omega_{\text{const}} = dx_1 \wedge dx_2 \wedge dx_3 + dx_4 \wedge dx_5 \wedge dx_6$. Show that ω_{const} is closed and that ω and ω_{const} agree at $0 \in \mathbb{R}^6$.
4. Set $\tilde{\omega} = t\omega + (1-t)\omega_{\text{const}}$. Calculate $\dot{\tilde{\omega}}$. Show that it is exact and find a potential α of the form $\alpha = h dx_5 \wedge dx_6$ such that $\alpha_0 = 0$.
5. Find a time-dependent vector field X near the origin such that $\iota_X \tilde{\omega} = -\alpha$.
6. Argue why this implies that near the origin there is a diffeomorphism ϕ with $\phi^*\omega = \omega_{\text{const}}$.

Exercise 6. Let $\mathbf{FinSet}$ be the category of finite sets with all maps as morphisms, and let $\mathcal{C}$ be its subcategory with all finite sets as objects and all bijections between them as morphisms. Denote by $i : \mathcal{C} \rightarrow \mathbf{FinSet}$ the inclusion functor. Determine whether i is

- a) fully faithful,
- b) essentially surjective,
- c) an equivalence of categories.